Introduction to $\mathbb{A}^1$-enumerative geometry via $\mathbb{A}^1$-degree and applications

$\mathbb{A}^1$-enumerative geometry via $\mathbb{A}^1$-degree

(Kass-Wickelgren)

Motivation from classical topology:

$$\deg : [S^n, S^n] \longrightarrow \mathbb{Z}$$

$$\begin{matrix} H_n(S^n) & \longrightarrow & H_n(S^n) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \end{matrix}$$

$f: S^n \to S^n$ $\quad p \in S^n$ regular value

$f^{-1}(p) = \{q_1, \dots, q_m\}$

then $\quad \deg f = \sum_{q \in f^{-1}(p)} \deg_q f$

local degree

V small ball around p $\qquad f^{-1}(p) \cap U = \{q\}$

U — " — q

$$\bar{f} : S^n \simeq U/\partial U \longrightarrow V/\partial V \simeq S^n$$

$$\begin{matrix} U/\partial U & & V/\partial V \\ \cong & & \cong \\ U/U - \{q\} & & V/V - \{p\} \end{matrix}$$

$\deg_q f := \deg \bar{f} \in \{\pm 1\}$ since $\bar{f}$ homeo formula from differential topology

$$\begin{array}{ccc} T_q f : & T_q S^n \to & T_p S^n \\ \parallel & \shortparallel & \shortparallel \\ (f_1, \dots, f_n) & \mathbb{R}^n & \mathbb{R}^n \end{array}$$

$$J(q) := \det \frac{\partial f_i}{\partial x_j}$$

Then $\deg_q f = \begin{cases} +1 & \text{if } J(q) > 0 \\ -1 & \text{if } J(q) < 0 \end{cases}$

want to do this over any field k
not only $\mathbb{R}$

Tool: $\mathbb{A}^1$-homotopy theory
and Morel's $\mathbb{A}^1$-degree

htpy theory on smooth schemes$/k$

$$[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to GW(k)$$

Note that $\mathbb{P}^n/\mathbb{P}^{n-1}(\mathbb{R}) \simeq S^n$

Need:
- quotients
- $\mathbb{A}^1$-htpy classes
- $GW(k)$

Crash course in $\mathbb{A}^1$-homotopy theory

Start with Sm_k = smooth schemes$/k$
(separated of finite type)

$$Sm_k \xrightarrow{\text{Yoneda}} sPre(Sm_k) \qquad K \mapsto (U \mapsto K)$$
$$X \mapsto Map(-,X) \qquad \text{discrete sSet} \qquad \text{constant presheaf}$$

closed under finite limits and colimits

$\Rightarrow$ Can make sense of

$$\operatorname{colim}\left(\begin{tikzcd} \mathbb{P}^{n-1}_k \arrow[r] \arrow[d] & \mathbb{P}^n_k \\ * & \end{tikzcd}\right) = \mathbb{P}^n_k / \mathbb{P}^{n-1}_k$$

$sPre(Sm_k)$ = simplicial model cat

or ∞-cat

$\swarrow$ has notion of weak equivalence

$\searrow$ has an associated homotopy category

Bousfield localization imposes additional weak equalences

$$Sm_k \to sPre(Sm_k) \xrightarrow[L_{Nis}]{} Sh_k \xrightarrow[L_{\mathbb{A}^1}]{} Spc_k$$

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl] \\ U \arrow[r] \arrow[ur, "p"] & X \end{tikzcd}$$

$X \times \mathbb{A}^1 \to X$ is weak eq

$[\ ,\ ]_{\mathbb{A}^1}$ = maps in $ho(Spc_k)$

$\uparrow$ htpy category

Morel's degree:

$$\left[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n/\mathbb{P}^{n-1}_k\right]_{\mathbb{A}^1} \longrightarrow GW(k)$$

$\uparrow$

iso for $n>1$
epi for $n=1$

$GW(k)$ = Grothendieck-Witt ring of k
= group completion of semi-ring of isometry classes of non-degenerate bilinear symmetric forms

generators: $\langle a \rangle \quad a \in k^*$

$\langle a \rangle = \left((x,y) \mapsto axy \right)$

relations:
1) $\langle a \rangle = \langle ab^2 \rangle$
2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$
3) $\langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle$
(4) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$)

Ex: · $GW(\mathbb{C}) \underset{rk}{\cong} \mathbb{Z}$

· $GW(\mathbb{R}) \underset{(rk, sgn)}{\cong} \mathbb{Z} \times \mathbb{Z}$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

$sgn = \#1 - \#-1$

$\cdot\ GW(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$

$\uparrow$

(rk, disc)

$\shortparallel$

det matrix

$\mathbb{A}^1$-local degree

$\deg : [S^n, S^n] \longrightarrow \mathbb{Z} \rightsquigarrow \deg^{\mathbb{A}^1} : \left[\mathbb{P}^n_k / \mathbb{P}^{n-1}_k , \mathbb{P}^n_k / \mathbb{P}^{n-1}_k\right]_{\mathbb{A}^1} \to GW(k)$

$f : \mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/\mathbb{P}^{n-1}$ $\rightsquigarrow$

$f : S^n \to S^n$ $\quad p \in S^n$ regular value

$f^{-1}(p) = \{q_1, \dots, q_m\}$

then $\quad \deg f = \sum_{q \in f^{-1}(p)} \deg_q f$

local degree

V ~~small~~ ball around p $\qquad f^{-1}(p) \cap U$

U — " — q $\qquad = \{q\}$

$\bar{f} : S^n \simeq U/\partial U \longrightarrow V/\partial V \simeq S^n$

$\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{A}^n/\mathbb{A}^n - 0 \overset{\text{Nis}}{\underset{\text{coord}}{\simeq}} U/U-\{q\}$ $\qquad$ $V/V-\{p\} \overset{\text{Nis}}{\underset{\text{coord}}{\simeq}} \mathbb{A}^n/\mathbb{A}^n - 0 \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$

$\deg_q f := \deg \bar{f} \quad \in \{\pm 1\}$ since $\bar{f}$ homeo

$\rightsquigarrow \bar{f} : \mathbb{A}^n / \mathbb{A}^n - \{q\} \longrightarrow \mathbb{A}^n / \mathbb{A}^n - \{p\}$

$\quad \simeq \mathbb{P}^n/\mathbb{P}^{n-1} \qquad\qquad \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$

$$\deg_q f := \deg \bar{f}$$

Computation:

$$T_q f : T_q S^n \to T_p S^n \rightsquigarrow \mathbb{A}^n \to \mathbb{A}^n$$

$$\begin{matrix} \| & \| & \| \\ (f_1, \dots, f_n) & \mathbb{R}^n & \mathbb{R}^n \end{matrix}$$

$$J(q) := \det \frac{\partial f_i}{\partial x_j}$$

Then $\deg_q f = \begin{cases} +1 & \text{if } J(q) > 0 \\ -1 & \text{if } J(q) < 0 \end{cases}$

$$\deg_q f := \langle J(q) \rangle$$

if $J(q) \neq 0$

If q not defined over k

then $\mathrm{Tr}_{k(q)/k}(\langle J(q) \rangle)$

$\mathrm{Tr}_{L/k} : GW(L) \to GW(k)$

L/k $\quad V \times V \xrightarrow{B} L \longmapsto \left(V \times V \xrightarrow{B} L \xrightarrow{\mathrm{Tr}_{L/k}} k \right)$

If $J(q)=0 \rightsquigarrow$ EKL-form

Eisenbud-Khimshiashvili-Levine

$f: \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero at the origin

local algebra $Q := Q_0(f) = \dfrac{k[x_1,\dots,x_n]_0}{(f_1,\dots,f_n)}$

$E := \det a_{ij}$ for $f_i(x) = \underset{0}{f_i(0)} + \sum a_{ij} x_j$

=: distinguished socle element

socle of a ring = sum of minimal nonzero ideals

here E generates socle

Let $\phi: Q \to k$ st $\phi(E) = 1$

The EKL-form of f at zero is the class of $B_\phi : Q \times Q \to k$

$B_\phi(a,b) = \phi(ab)$

in $GW(k)$

Kass-Wickelgren show this is well-defined

Example: $x^m = f : \mathbb{A}^1 \to \mathbb{A}^1$

$$Q = \frac{k[x]_{(x)}}{(x^m)} \qquad E = x^{m-1}$$

	1	x	$\ldots$	$x^{m-1} = E$
1	0	0	$\ldots$ 0	1
x	0	0	$\ldots$ 1	
$\vdots$	$\vdots$			
$\vdots$	0	1		0
x^{m-1}	1			

$\mathbb{H} = \langle 1 \rangle + \langle -1 \rangle$

$$\rightsquigarrow \begin{cases} \frac{m}{2}\mathbb{H} & \text{for } m \text{ even} \\ \frac{m-1}{2}\mathbb{H} + \langle 1 \rangle & m \text{ odd} \end{cases}$$

Example 2: $f : \mathbb{A}^2 \to \mathbb{A}^2$

$(x^2 - y^2, 2xy)$

$\boxed{f(z) = z^2}$

$$E = \det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 = 2x^2$$

	1	x	y	x^2+y^2
1	0	0	0	1
x	0	$\frac{1}{2}$	0	
y	0	0	$\frac{1}{2}$	0
x^2+y^2	1	0	0	0

$\rightsquigarrow \langle \tfrac{1}{2} \rangle + \langle \tfrac{1}{2} \rangle$

$+ \mathbb{H}$

Counting lines

The $\mathbb{A}^1$- Euler number

$\pi : \mathcal{E} \to X$ VB rk r dim $X = r$

Def $\pi : \mathcal{E} \to X$ is *relatively oriented* if $\mathrm{Hom}(\det TX, \det \mathcal{E}) \cong L^{\otimes 2}$

Def $\phi : U \to \mathbb{A}^r$ étale, $U \ni p$, $U \subset X$, which induces iso on $\kappa(p)$ $=:$ *Nisnevich coordinates around p*

Def A trivialization of $\mathcal{E}|_U$ is *compatible* with ϕ and the relative orientation if

$$\mathrm{Hom}(\det TX|_U, \det \mathcal{E}|_U) = \text{square}$$

distinguished basis $\longmapsto$ distinguished basis

Let $\sigma : X \to \mathcal{E}$ be a section with only isolated zeros

Let q be an isolated zero

$\phi : U \to \mathbb{A}^r$ Nis coord compatible with relative orientation

$\rightsquigarrow$ around q σ looks like

$$f : \underset{\substack{\cup \\ q = 0}}{\mathbb{A}^n} \to \mathbb{A}^r$$

Let $\mathrm{ind}(q) := \deg f$

Def The $\mathbb{A}^1$-Euler number $e(\mathcal{E}, \sigma)$ $\pi : \mathcal{E} \to X$ is

$$\sum_{q \in \sigma^{-1}(0)} \mathrm{ind}_q$$

Thm (Kass-Wickelgren): with only isolated zeros

If any 2 sections σ and σ' of $\mathcal{E}$ can be connected by an $\mathbb{A}^1$ then $e(\mathcal{E}, \sigma)$ is independent of σ.

Reason: $GW(k) \cong GW(k[t])$

Application:

1) Counting lines on cubic surfaces

$$X = \{f = 0\} \subseteq \mathbb{P}^3$$

homogeneous of degree 3

$\rightsquigarrow$ section $\sigma_f : Gr(2,4) \to Sym^3 S^*$ by restriction

$Gr(2,4)$ = lines in $\mathbb{P}^3$, S = tautological bundle

$\dim Gr(2,4) = 4$

$\operatorname{rk} Sym^3 S^* = 4$

Macaulay 2 $\Rightarrow$ rk 27 disc 1 sgn 3

$\rightsquigarrow 15\langle 1\rangle + 12\langle -1\rangle \in GW(k)$

2) Lines on a quintic 3-fold

$X = \{f = 0\} \subseteq \mathbb{P}^4$

$\uparrow$ homogeneous degree 5

$\rightsquigarrow$ section $\sigma_f : Gr(2,5) \to Sym^5 S^*$

dim 6   rk 6

Problem: too complicated for my computer

Albano-Katz:

On $\{F: X_0^5 + \ldots + X_4^5 = 0\} \subseteq \mathbb{P}^4$

there are infinitely many lines

namely those s,t coordinates

$(s : -\zeta s : at : bt : ct)$ $a^5 + b^5 + c^5 = 0$

ζ = 5th rooth of unity

so $\sigma_F^{-1}(0) = \bigcup_{50} C$ $\leftarrow$ 1 dim

Thm (Albano-Katz): There are 2875 distinguished complex lines on X that deform with

$$X_t = \{ F + tF_1 + t^2 F_2 + \ldots = 0 \} \subseteq \mathbb{P}^4$$

1) lines in the intersection of 2 components $\ell = (s:-\zeta s : t:-\zeta' t : 0)$ deform with multiplicity 5

2) in each component there are 10 lines which deform with multiplicity 2

In total $50 \cdot 10 \cdot 2 + 375 \cdot 5 = 2875$

$$375 = \frac{\binom{5}{2} \cdot \binom{3}{2} \cdot 5 \cdot 5}{2} = 15 \cdot 5 \cdot 5$$

Thm (P.) $\operatorname{ind}(\ell_t)$ does not depend on deformation

reason: $GW(k) = GW(k[t])$

deformation of a distinguished line

Compute $\sum \operatorname{ind}(\ell_t) \in GW(k((t)))$

$$\rightsquigarrow 50 \cdot 10 \cdot \mathbb{H} + 15(2\mathbb{H} + \langle 1 \rangle) + 90\, \mathrm{Tr}_{k(5)/k}(2\mathbb{H} + \langle 1 \rangle)$$

$$= 1340\, \mathbb{H} + 90(\langle 1 \rangle + \langle -5 \rangle) + 15 \langle 1 \rangle$$

$$\sim 1445 \langle 1 \rangle + 1430 \langle -1 \rangle \quad \text{for char } k \neq 5$$

Q: What geometric information does $\mathrm{ind}\,[\ell]$ give?

cubic surfaces (Segre, over $\mathbb{R}$; Kass-Wickelgren, over k, char $k \neq 2$)

$\ell \subseteq X \subseteq \mathbb{P}^3$ cubic surface

Gauß map

$$\ell \cong \mathbb{P}^1 \xrightarrow{\deg 2} \mathbb{P}^1 = \text{2-planes in } \mathbb{P}^3 \text{ containing } \ell$$

$$p \longmapsto T_pX$$

for a $p \in \ell$ $\exists!$ $q \in \ell$ with $T_pX = T_qX$

$\rightsquigarrow$ involution $i: \ell \to \ell$ sending p to q

fixed pts of i are defined over $k(\sqrt{\alpha})$ $\alpha \in k^\times/(k^\times)^2$

Call $\langle \alpha \rangle \in GW(k)$ the type of ℓ

Thm: $\mathrm{Type}(\ell) = \mathrm{ind}\,[\ell] \in GW(k)$

Ex: over $\mathbb{R}$ there are 2 types

Quintic 3-folds (Finashin–Kharlamov, P.)
over $\mathbb{R}$ — over k

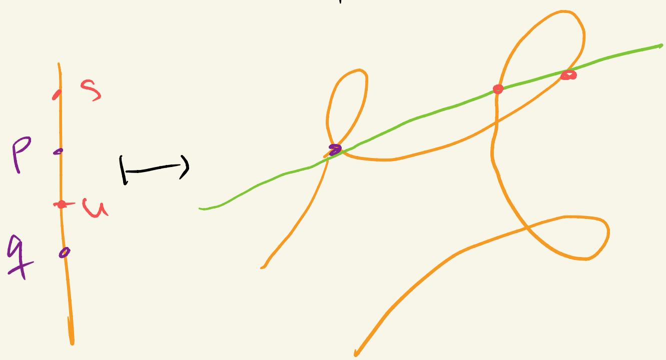
$\ell \in X \subseteq \mathbb{P}^4$
quintic 3-fold

Gauß map

$\ell \cong \mathbb{P}^1 \xrightarrow{\deg 4} \mathbb{P}^2 =$ 3-planes in $\mathbb{P}^4$ containing ℓ

$p \longmapsto T_p X$

- ∃ 3 pairs of pts on ℓ with the same tangent space in X
- Let p, q be such a pair and let $T = T_p X = T_q X$ then to $s \in \ell$ ∃! $u \in \ell$ st $T \cap T_s X = T \cap T_u X$



Def $\mathrm{Type}(\ell) := \prod \langle \alpha_j \rangle \in GW(k)$

⇝ 3 involutions with fixed pts defined over $F_1(\sqrt{\alpha_1}), F_2(\sqrt{\alpha_2}), F_2(\sqrt{\alpha_3})$

Thm (P): $\mathrm{Type}(\ell) = \mathrm{ind}(\ell)$

So $$\sum_{\ell \subseteq X} \mathrm{Tr}_{k(\ell)/k}(\mathrm{Type}(\ell))$$

cubic surface

$= 15\langle 1\rangle + 12\langle -1\rangle \in GW(k)$

and

$$\sum_{\ell \subseteq X} \mathrm{Tr}_{k(\ell)/k}(\mathrm{Type}(\ell))$$

quintic 3-fold

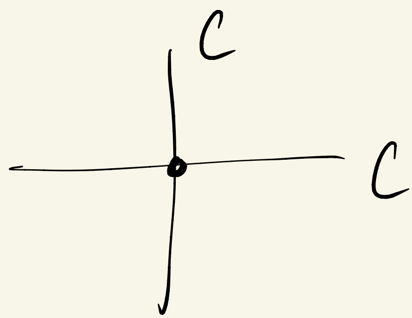
$= 1445\langle 1\rangle + 1430\langle -1\rangle \in GW(k)$

$$X_0^5 + \dots + X_4^5 = 0$$

$$(s : -s : t : -t : 0)$$

local analytic

$$\frac{k[x,y]}{(x^3y^2,\ x^2y^3)}$$



$$\frac{k((t))[x,y]}{\big(\underbrace{x^3y^2 + t^5 f_1 + t^{10} f_2 + \dots}_{f_t},\ \underbrace{x^2y^3 + t^5 g_1 + t^{10} g_2 + \dots}_{g_t}\big)}$$

There is a solution

(x_t, y_t) $\qquad f_t(x_t, y_t) = 0 = g_t(x_t, y_t)$

$x_t = t x_1 + t^2 x_2 + \dots$

$y_t = t y_1 + t^2 y_2 + \dots$

$t^5: \ x_1^3 y_1^2 = -f_1(0) = a$

$x_1^2 y_1^3 = -g_1(0) = b$

$$\frac{x_1}{y_1} = \frac{a}{b}$$

$$GW(k((t)))$$

$$GW(\cancel{k[[t]]}) \cong GW(k)$$

$$\rightsquigarrow \quad x_1 = \sqrt[5]{\frac{a^3}{b^2}}$$

$$y_1 = \sqrt[5]{\frac{a^2}{b^3}}$$

$$k(\sqrt[5]{ab})$$

$$\parallel$$

$$L$$

$$\deg_{(x_t, y_t)}(f_t, g_t) = ?$$

$$\det \begin{pmatrix} 3x_1^2y_1^2t^4 + \dots & 2x_1y_1^3t^4 + \dots \\ 2x_1^3y_1t^4 + & 3x_1^2y_1^2t^4 + \dots \end{pmatrix}$$

$$= \cancel{t^8 x_1^4 y_1^4} \cdot 5$$

$$\mathrm{Tr}_{L/k}\langle 5\rangle = 2\mathbb{H} + \langle 1\rangle$$

$$(x^3y^2 + t^5 f_1 + t^{10} f_2 + \dots,\ x^2y^3 + t^5 g_1 + t^{10} g_2 + \dots)$$

Multiplicity 2 lines

$$\frac{k[x,y]}{(x, y^2)}$$

$$L = k(\sqrt{d})$$

$$\mathrm{Tr}_{k(\sqrt{d})/k}(\langle \sqrt{d} \rangle) = H$$